

FACTORIZATION IN GENERALIZED CALOGERO-MOSER SPACES

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ABSTRACT. Using a recent construction of Bezrukavnikov and Etingof, [BE], we prove that there is a factorization of the Etingof-Ginzburg sheaf on the generalized Calogero-Moser space associated to a complex reflection group. In the case $W = S_n$, this confirms a conjecture of Etingof and Ginzburg, [EG].

1. INTRODUCTION

In this paper we apply a recent construction of Bezrukavnikov and Etingof, [BE], to the study of the centres of the rational Cherednik algebras at $t = 0$. The affine varieties $X_{\mathbf{c}}(W)$, corresponding to these centres are called the *generalized Calogero-Moser spaces* and are known to influence the representation theory of the algebras. We show that there exists an isomorphism of schemes

$$(1) \quad \Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0),$$

where $\pi_W : X_{\mathbf{c}}(W) \rightarrow \mathfrak{h}/W$ and W_b is a parabolic subgroup of W associated to the orbit $b \in \mathfrak{h}/W$ (for precise definitions see Section 2). In order to relate the representation theory of the algebras $H_{0,\mathbf{c}}$ to the varieties $X_{\mathbf{c}}$, Etingof and Ginzburg introduced the coherent sheaf $\mathcal{R}[W]$, defined by $\Gamma(X_{\mathbf{c}}, \mathcal{R}[W]) = H_{0,\mathbf{c}}\mathbf{e}$. Our main result describes the pushforward of $\mathcal{R}[W]_{|\pi_W^{-1}(b)}$ by Φ .

Theorem. On $\pi_{W_b}^{-1}(0)$ there is an isomorphism of W -equivariant sheaves

$$\Phi_* \left(\mathcal{R}[W]_{|\pi_W^{-1}(b)} \right) \simeq \text{Ind}_{W_b}^W \mathcal{R}[W_b]_{|\pi_{W_b}^{-1}(0)}$$

In particular, this theorem proves that the sheaf $\mathcal{R}[W]$ on the Calogero-Moser space associated to the symmetric group factorizes as conjectured by Etingof and Ginzburg, [EG, page 319].

2. THE RATIONAL CHEREDNIK ALGEBRA

2.1. Definitions and notation. Let W be a complex reflection group, \mathfrak{h} its reflection representation over \mathbb{C} with rank $\mathfrak{h} = n$, and \mathcal{S} the set of all complex reflections in W . The idempotent in $\mathbb{C}W$ corresponding to the trivial representation will be denoted \mathbf{e}_W . Let $(\cdot, \cdot) : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$ be the natural pairing defined by $(y, x) = x(y)$. For $s \in \mathcal{S}$, fix $\alpha_s \in \mathfrak{h}^*$ to be a basis of the one dimensional space $\text{Im}(s - 1)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathfrak{h}$ a basis of the one dimensional space $\text{Im}(s - 1)|_{\mathfrak{h}}$ such that $\alpha_s(\alpha_s^\vee) = 2$. Choose $\mathbf{c} : \mathcal{S} \rightarrow \mathbb{C}$ to be a W -invariant function and t a complex number. The *rational Cherednik algebra*, $H_{t,\mathbf{c}}(W)$, as introduced by Etingof and Ginzburg, [EG, page 250], is the quotient of the skew group algebra of the tensor algebra, $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, by the ideal generated by the relations

$$(2) \quad [x_1, x_2] = 0 \quad [y_1, y_2] = 0 \quad [x_1, y_1] = t(y_1, x_1) - \sum_{s \in S} \mathbf{c}(s)(y_1, \alpha_s)(\alpha_s^\vee, x_1)s$$

$\forall x_1, x_2 \in \mathfrak{h}^*$ and $y_1, y_2 \in \mathfrak{h}$.

Since there is an isomorphism $H_{\lambda t, \lambda \mathbf{c}}(W) \cong H_{t, \mathbf{c}}(W)$ for any $\lambda \in \mathbb{C}^*$, we can restrict ourselves to considering the cases $t = 0$ or 1 .

2.1.1. Parabolic subgroups. For a point $b \in \mathfrak{h}$, the stabilizer subgroup of W with respect to b will be denoted W_b . By a theorem of Steinberg, [S2, Theorem 1.5], W_b is itself a complex reflection group. If $(\mathfrak{h}^{*W_b})^\perp$ denotes the vector subspace of \mathfrak{h} consisting of all vectors y in \mathfrak{h} such that $x(y) = 0 \quad \forall y \in \mathfrak{h}^{*W_b}$ then $\mathfrak{h} = \mathfrak{h}^{W_b} \oplus (\mathfrak{h}^{*W_b})^\perp$ is a decomposition of \mathfrak{h} as a W_b -module. Note that $(\mathfrak{h}^{*W_b})^\perp$ is a faithful reflection representation of W_b of minimal rank.

2.1.2. Centralizer algebras. We recall the centralizer construction described in [BE, 3.2]. Let A be a \mathbb{C} -algebra equipped with a homomorphism $H \longrightarrow A^\times$, where H is a finite group. Let G be another finite group such that H is a subgroup of G . The algebra $C(G, H, A)$ is defined to be the centralizer of A in the right A -module $P := \text{Fun}_H(G, A)$ of H -invariant, A -valued functions on G . By making a choice of left coset representatives of H in G , $C(G, H, A)$ is realized as the algebra of $|G/H|$ by $|G/H|$ matrices over A . For $w, g \in G$ and $f \in \text{Fun}_H(G, A)$, $w \cdot f(g) := f(gw)$ defines, by linearity, an embedding $\iota : \mathbb{C}G \hookrightarrow C(G, H, A)$. Let $\mathbf{e}_G \in \mathbb{C}G$ and $\mathbf{e}_H \in \mathbb{C}H$ denote the idempotents corresponding to the trivial representation of G and H respectively, where $\mathbb{C}H$ is considered as a subalgebra of A .

Lemma 2.1. *There are isomorphisms of $\mathbb{C}G$ - $Z(A)$ -bimodules*

$$C(G, H, A) \cdot \iota(\mathbf{e}_G) \simeq \text{Fun}_H(G, A\mathbf{e}_H) \simeq \text{Ind}_H^G A\mathbf{e}_H,$$

where $Z(A)$ denotes the centre of A . Here $\mathbb{C}G$ acts on $C(G, H, A)$ by multiplication on the left via ι and on the left of $\text{Fun}_H(G, A\mathbf{e}_H)$ also via ι .

Proof. The second isomorphism is clear from the definition of $\text{Fun}_H(G, A)$. Let $\delta \in \text{Fun}_H(G, A)$ be the function defined by $\delta(g) = \mathbf{e}_H$, for all $g \in G$. We define a linear map ζ from $C(G, H, A) \cdot \iota(\mathbf{e}_G)$ to $\text{Fun}_H(G, A\mathbf{e}_H)$ and a map η in the opposite direction by

$$\begin{aligned} \zeta : \quad M \cdot \iota(\mathbf{e}_G) &\mapsto M(\delta) \\ \eta : \quad f &\mapsto \left(h(-) \mapsto f(-) \sum_{g \in G} h(g) \right), \end{aligned}$$

where $M \in C(G, H, A)$, $f \in \text{Fun}_H(G, A\mathbf{e}_H)$ and $h \in \text{Fun}_H(G, A)$. After fixing left coset representatives of H in G , a direct calculation shows that η is both a left and right inverse to ζ . The G -equivariance of ζ is clear since

$$g \cdot \zeta(M\iota(\mathbf{e}_G)) = g \cdot M(\delta) = \iota(g)(M(\delta)) = (\iota(g)M)(\delta) = \zeta(g \cdot M\iota(\mathbf{e}_G))$$

The $Z(A)$ -equivariance of ζ is similarly clear. □

2.2. Completing the rational Cherednik algebra. For each point $b \in \mathfrak{h}$, the completion, $\widehat{H}_{1,\mathbf{c}}(W)_b$, at the orbit $W \cdot b \in \mathfrak{h}/W$ of $H_{1,\mathbf{c}}(W)$ is defined in [BE, 2.4]. However, the notion of completion at $W \cdot b$ works equally well when $t = 0$ because $H_{0,\mathbf{c}}(W)$ can be thought of as a sheaf of algebras on the affine variety \mathfrak{h}/W . Therefore, if $\mathbb{C}[[\mathfrak{h}/W]]_b$ denotes the completion of $\mathbb{C}[\mathfrak{h}/W]$ at $W \cdot b$, we define the completion of $H_{0,\mathbf{c}}(W)$ at b to be

$$(3) \quad \widehat{H}_{0,\mathbf{c}}(W)_b := \mathbb{C}[[\mathfrak{h}/W]]_b \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_{0,\mathbf{c}}(W)$$

Crucially, we note that [BE, Theorem 3.2] is independent of the parameter t and hence can be applied to the case $t = 0$. We state it here for completeness.

Theorem 2.2 ([BE], Theorem 3.2). *Let $b \in \mathfrak{h}$, and define \mathbf{c}' to be the restriction of \mathbf{c} to the set S_b of reflections in W_b . Then one has an isomorphism*

$$(4) \quad \theta : \widehat{H}_{t,\mathbf{c}}(W, \mathfrak{h})_b \rightarrow C(W, W_b, \widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0),$$

defined by the following formulas. Suppose that $f \in P = \text{Fun}_{W_b}(W, \widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0)$. Then

$$(\theta(u)f)(w) = f(wu), u \in W;$$

for any $\alpha \in \mathfrak{h}^$,*

$$(\theta(x_\alpha)f)(w) = (x_{w\alpha}^{(b)} + (w\alpha, b))f(w),$$

where $x_\alpha \in \mathfrak{h}^ \subset H_{t,\mathbf{c}}(W, \mathfrak{h})$, $x_{w\alpha}^{(b)} \in H_{t,\mathbf{c}'}(W_b, \mathfrak{h})$; and for any $a \in \mathfrak{h}$,*

$$(\theta(y_a)f)(w) = y_{wa}^{(b)}f(w) + \sum_{s \in S: s \notin W_b} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(wa)}{x_{\alpha_s}^{(b)} + \alpha_s(b)} (f(sw) - f(w)).$$

where $y_a \in \mathfrak{h} \subset H_{t,\mathbf{c}}(W, \mathfrak{h})$ and $y_a^{(b)}$ the same vector considered now as an element of $H_{t,\mathbf{c}'}(W_b, \mathfrak{h})$.

3. THE ETINGOF-GINZBURG SHEAF

Let $Z_{\mathbf{c}}(W)$ denote the centre of $H_{0,\mathbf{c}}(W)$ and $X_{\mathbf{c}}(W) = \text{maxspec}(Z_{\mathbf{c}}(W))$, the corresponding affine variety. The space $X_{\mathbf{c}}(W)$ is called the *generalized Calogero-Moser space* associated to the complex reflection group W at the parameter \mathbf{c} . For $b \in \mathfrak{h}$, the maximal ideal of $\mathbb{C}[\mathfrak{h}/W]$ corresponding to $W \cdot b$ will be written $\mathfrak{m}(b)$ and the two-sided ideal of $H_{0,\mathbf{c}}(W)$ generated by the elements of $\mathfrak{m}(b)$ will be denoted $\langle \mathfrak{m}(b) \rangle$. Similarly, the maximal ideal of $\mathbb{C}[\mathfrak{h}/W_b]$ corresponding to $W_b \cdot p$, $p \in \mathfrak{h}$, will be written $\mathfrak{n}(p)$.

Let \mathcal{A} denote the set of reflecting hyperplanes of W in \mathfrak{h} and, for each $H \in \mathcal{A}$, let $L_H \in \mathfrak{h}^*$ be a linear functional whose kernel is H (e.g. $\alpha_s \in \mathfrak{h}^*$ if s is a reflection about H). Choose homogeneous algebraically independent generators F_1, \dots, F_n of $\mathbb{C}[\mathfrak{h}]^W$ and P_1, \dots, P_n of $\mathbb{C}[\mathfrak{h}]^{W_b}$. The following description of the Jacobian is due to Steinberg, [S1, Lemma].

$$(5) \quad \Pi_W := \det \left(\frac{\partial F_i}{\partial x_j} \right) = k \prod_{H \in \mathcal{A}} L_H^{e_H - 1}$$

where e_H is the order of the cyclic group W_H of elements of W that fix H pointwise and k a non-zero scalar.

Lemma 3.1. *For each $b \in \mathfrak{h}$ the map $\Psi : \mathbb{C}[[\mathfrak{h}/W_b]]_0 \longrightarrow \mathbb{C}[[\mathfrak{h}/W_b]]_0$ defined by*

$$P_i(\mathbf{x}) \mapsto F_i(\mathbf{x} + b) - F_i(b)$$

is an automorphism.

Proof. Since $F_i(\mathbf{x} + b) - F_i(b) \in \mathfrak{n}(0)$ for all i there exist polynomials Q_1, \dots, Q_n such that $F_i(\mathbf{x} + b) - F_i(b) = Q_i(P_1, \dots, P_n)$. The chain rule gives

$$D := \det \left(\frac{\partial(F_i(\mathbf{x} + b) - F_i(b))}{\partial x_j} \right) = \det \left(\frac{\partial Q_i}{\partial P_k} \right) \det \left(\frac{\partial P_k}{\partial x_j} \right)$$

However, $D = \Pi_W(\mathbf{x} + b)$ and this gives

$$\prod_{H \in \mathcal{A}} L_H^{e_H - 1}(\mathbf{x} + b) = \det \left(\frac{\partial Q_i}{\partial P_k} \right) \prod_{H \in \mathcal{A} \text{ with } b \in H} L_H^{e_H - 1}(\mathbf{x})$$

Since $L_H(\mathbf{x} + b) = L_H(\mathbf{x})$ if and only if $b \in H$, we get

$$\det \left(\frac{\partial Q_i}{\partial P_k} \right) = \prod_{H \in \mathcal{A} \text{ with } b \notin H} L_H^{e_H - 1}(\mathbf{x} + b)$$

and

$$\det \left(\frac{\partial Q_i}{\partial P_k} \right) (0) = \prod_{H \in \mathcal{A} \text{ with } b \notin H} L_H^{e_H - 1}(b) \neq 0.$$

Hence, by [E, Exercise 7.25], Ψ is an isomorphism. \square

As a consequence of Theorem 2.2, we have an isomorphism of quotient algebras.

Corollary 3.2. *Let $\theta : \widehat{H}_{0,\mathbf{c}}(W, \mathfrak{h})_b \rightarrow C(W, W_b, \widehat{H}_{0,\mathbf{c}'}(W_b, \mathfrak{h})_0)$ be the isomorphism (4). Then θ descends to an isomorphism*

$$(6) \quad \theta : \frac{H_{0,\mathbf{c}}(W, \mathfrak{h})}{\langle \mathfrak{m}(b) \rangle} \xrightarrow{\sim} C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right).$$

Proof. For $a \in \mathfrak{h}$, $\alpha \in \mathfrak{h}^*$ and $w \in W$, $(x_{w \cdot \alpha} + (w\alpha, b))(a) = (w\alpha, a) + (w\alpha, b) = (w \cdot x_\alpha)(a + b)$. Therefore $\theta(g)(f(w)) = (w \cdot g)(\mathbf{x} + b)f(w) = g(\mathbf{x} + b)f(w)$ for all $g \in \mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[[\mathfrak{h}]]_b$ and $f \in \text{Fun}_{W_b}(W, \widehat{H}_c(W_b, \mathfrak{h})_0)$. Now choose $u \in W_b$, then

$$u \cdot g(\mathbf{x} + b) = g(u^{-1} \cdot \mathbf{x} + b) = g(u^{-1} \cdot (\mathbf{x} + b)) = g(\mathbf{x} + b)$$

shows that $g(\mathbf{x} + b) \in \mathbb{C}[\mathfrak{h}]^{W_b}$. Hence, if $g \in \mathfrak{m}(b) \triangleleft \mathbb{C}[\mathfrak{h}]^W$, then $g(\mathbf{x} + b) \in \mathfrak{n}(0) \triangleleft \mathbb{C}[\mathfrak{h}]^{W_b}$. This shows that $\theta(g)f(w) \in \mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b)_0$ and

$$(7) \quad \theta(\mathfrak{m}(b)\widehat{H}_{t,\mathbf{c}}(W, \mathfrak{h})_b) \subseteq C(W, W_b, \mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0).$$

The ideal $\mathfrak{m}(b)$ in $\mathbb{C}[\mathfrak{h}]^W$ is generated by $F_1(\mathbf{x}) - F_1(b), \dots, F_n(\mathbf{x}) - F_n(b)$ and we have $\theta(F_i(\mathbf{x}) - F_i(b))f(w) = (F_i(\mathbf{x} + b) - F_i(b))f(w)$. The statement of Lemma 3.1 is equivalent to the fact that

$$\{F_1(\mathbf{x} + b) - F_1(b), \dots, F_n(\mathbf{x} + b) - F_n(b)\} \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0 = \mathfrak{n}(0)\mathbb{C}[[\mathfrak{h}/W_b]]_0.$$

This, together with (7), implies that

$$\theta(\mathfrak{m}(b)\widehat{H}_{t,c}(W, \mathfrak{h})_b) = C(W, W_b, \mathfrak{n}(0)\widehat{H}_{t,c'}(W_b, \mathfrak{h})_0).$$

and the isomorphism follows. \square

By [EG, Proposition 4.15], $\mathbb{C}[\mathfrak{h}]^W$ is contained in the centre of $H_{0,c}(W, \mathfrak{h})$ and the embedding defines a surjective morphism $\pi_W : X_c(W) \twoheadrightarrow \mathfrak{h}/W$. The algebra $Z_{0,c}(W)/\langle \mathfrak{m}(b) \rangle$ is the coordinate ring of the scheme-theoretic pull-back $\pi_W^{-1}(b)$. Comparing the centres of the algebras in Corollary 3.2 gives an isomorphism of (non-reduced) schemes.

Corollary 3.3. *For $b \in \mathfrak{h}$, there is a scheme-theoretic isomorphism*

$$(8) \quad \Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0)$$

Proof. The Satake isomorphism, [EG, Theorem 3.1], is the map $Z_{0,c}(W) \rightarrow \mathbf{e}_W H_{0,c}(W, \mathfrak{h}) \mathbf{e}_W$ defined by $z \mapsto z \mathbf{e}_W$. Since $\mathfrak{m}(b)H_{0,c}(W, \mathfrak{h})$ is a centrally generated ideal in $H_{0,c}(W, \mathfrak{h})$,

$$\mathfrak{m}(b)H_{0,c}(W, \mathfrak{h}) \cap \mathbf{e}_W H_{0,c}(W, \mathfrak{h}) \mathbf{e}_W = \langle \mathbf{e}_W \mathfrak{m}(b) \rangle,$$

where the right-hand side is considered as an ideal in $\mathbf{e}_W H_{0,c}(W, \mathfrak{h}) \mathbf{e}_W$. Therefore the Satake isomorphism descends to an isomorphism

$$(9) \quad S_{W,b} : \frac{Z_{0,c}(W)}{\mathfrak{m}(b)Z_{0,c}(W)} \xrightarrow{\sim} \mathbf{e}_W \left(\frac{H_{0,c}(W, \mathfrak{h})}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W.$$

As noted in [BE, Lemma 3.1 (ii)], the isomorphism (6) restricts to an isomorphism of subalgebras

$$(10) \quad \theta : \mathbf{e}_W \left(\frac{H_{0,c}(W, \mathfrak{h})}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W \xrightarrow{\sim} \mathbf{e}_{W_b} \left(\frac{H_{0,c'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \mathbf{e}_{W_b},$$

where $\theta(\mathbf{e}_W) = \mathbf{e}_{W_b}$. Here we have implicitly identified the spherical algebra on the right-hand side with a subalgebra of $C\left(W, W_b, \frac{H_{0,c'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle}\right)$. It is possible, though uninformative, to give an explicit description of this identification. Combining the isomorphisms of (9) and (10) produces the comorphism

$$(11) \quad (\Phi^*)^{-1} = S_{W_b,0}^{-1} \circ \theta \circ S_{W,b} : \frac{Z_{0,c}(W)}{\mathfrak{m}(b)Z_{0,c}(W)} \xrightarrow{\sim} \frac{Z_{0,c'}(W_b)}{\mathfrak{n}(0)Z_{0,c'}(W_b)}$$

corresponding to Φ . \square

Etingof and Ginzburg, [EG, page 247], introduced an important coherent sheaf on $X_c(W)$, which we now recall.

Definition 3.4. The *Etingof-Ginzburg sheaf* is the coherent sheaf $\mathcal{R}[W]$ on $X_c(W)$ corresponding to the finitely generated $Z_{0,c}(W)$ -module $H_{0,c}(W) \mathbf{e}_W$.

The coordinate ring of a Zariski-open subset $U \subseteq X_c(W)$ will be written $Z_{0,c}(W)_U$. We now conclude;

Theorem 3.5. *Let $\mathcal{R}[W]$ be the Etingof-Ginzburg sheaf on $X_{\mathbf{c}}(W)$ and $\mathcal{R}[W_b]$ the Etingof-Ginzburg sheaf on $X_{\mathbf{c}'}(W_b)$. For $b \in \mathfrak{h}/W$ we have an isomorphism of W -equivariant sheaves on $\pi_{W_b}^{-1}(0)$*

$$(12) \quad \Phi_* \left(\mathcal{R}[W]_{|_{\pi_{W_b}^{-1}(b)}} \right) \simeq \text{Ind}_{W_b}^W \mathcal{R}[W_b]_{|_{\pi_{W_b}^{-1}(0)}}.$$

Proof. Since $\pi_{W_b}^{-1}(b)$ is an affine scheme, to show that we have an isomorphism of W -equivariant sheaves as stated in (12) it suffices to show that the global sections are isomorphic as $(W, Z_{0,\mathbf{c}'}(W_b) / \langle \mathbf{n}(0) \rangle =: \mathbf{Z})$ -bimodules. Taking global sections gives

$$\Phi_* \left(\mathcal{R}[W]_{|_{\pi_{W_b}^{-1}(b)}} \right) (\pi_{W_b}^{-1}(0)) = \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathbf{m}(b) \rangle} \right) \mathbf{e}_W$$

and

$$\text{Ind}_{W_b}^W \mathcal{R}[W_b]_{|_{\pi_{W_b}^{-1}(0)}} (\pi_{W_b}^{-1}(0)) = \text{Ind}_{W_b}^W \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathbf{n}(0) \rangle} \right) \mathbf{e}_{W_b}.$$

Thus we must show that

$$\text{He} := \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathbf{m}(b) \rangle} \right) \mathbf{e}_W \simeq \text{Ind}_{W_b}^W \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathbf{n}(0) \rangle} \right) \mathbf{e}_{W_b}$$

as (W, \mathbf{Z}) -bimodules. Applying the isomorphism θ (of (6)) to He , and noting that the restriction of θ to $\mathbb{C}W$ is the map ι , gives

$$\theta : \text{He} \simeq C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathbf{n}(0) \rangle} \right) \iota(\mathbf{e}_W).$$

However, we now have two different actions of \mathbf{Z} on He . It acts on He , viewed as global sections, via the map Φ^* , but acts on the right of $C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathbf{n}(0) \rangle} \right) \iota(\mathbf{e}_W)$ via θ^{-1} . These two actions are the same: as stated in (11),

$$\Phi^* = S_{W,b}^{-1} \circ \theta^{-1} \circ S_{W_b,0},$$

therefore

$$h \mathbf{e}_W \cdot \Phi^*(z) = h \mathbf{e}_W \cdot S_{W,b}^{-1} \circ \theta \circ S_{W_b,0}(z) = h \mathbf{e}_W \cdot \mathbf{e}_W \theta^{-1}(\mathbf{e}_{W_b} \cdot z) = h \mathbf{e}_W \cdot \theta^{-1}(z),$$

where $z \in \mathbf{Z}$ and $h \mathbf{e}_W \in \text{He}$ (recall that $\theta(\mathbf{e}_W) = \mathbf{e}_{W_b}$, c.f. (11)). Noting that \mathbf{Z} is a subalgebra of the centre of $H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) / \langle \mathbf{n}(0) \rangle$, the required bimodule isomorphism is given by Lemma 2.1 where $G = W$, $H = W_b$ and $A = H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) / \langle \mathbf{n}(0) \rangle$. \square

Example 3.6. In the case $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$, the Calogero-Moser space $X_{\mathbf{c}}(S_n)$ has been shown by Etingof and Ginzburg, [EG, Theorem 1.23], to be isomorphic to the classical Calogero-Moser space as introduced by Kazhdan, Kostant and Sternberg and studied by Wilson, [W]. It is known to be smooth for $\mathbf{c} \neq 0$ ([EG, Corollary 16.2] or [W, Proposition 1.7]), therefore [EG, Theorem 1.7 (i)] implies that $\mathcal{R}[S_n]$ is a vector bundle of rank $n!$ on $X_{\mathbf{c}}(S_n)$. Identifying \mathbb{C}^n/S_n with $\text{Sym}^n(\mathbb{C})$, a point of \mathbb{C}^n/S_n has the form $n_1 x_1 + \dots + n_k x_k$, where $n_1 + \dots + n_k = n$ and $x_1, \dots, x_k \in \mathbb{C}$ are pairwise distinct. Given $b \in \mathbb{C}^n$ such that $S_n \cdot b = n_1 x_1 + \dots + n_k x_k$, the stabilizer $(S_n)_b$ is conjugate to $S_{n_1} \times \dots \times S_{n_k}$. For $W = S_n$, the isomorphism of Corollary 3.3 induces, after factoring out nilpotent elements, an isomorphism of varieties

$$(13) \quad \pi_{S_n}^{-1}(b) \simeq \pi_{S_{n_1}}^{-1}(0) \times \dots \times \pi_{S_{n_k}}^{-1}(0).$$

In [W, Lemma 7.1], Wilson explicitly constructs an isomorphism between the subvarieties of the classical Calogero-Moser space coinciding with the varieties of (13). Let \boxtimes denote the external tensor product of vector bundles, then Theorem 3.5 implies that

$$\Phi_* \left(\mathcal{R}[S_n]_{|\pi_{S_n}^{-1}(b)} \right) \simeq \text{Ind}_{S_{n_1} \times \dots \times S_{n_k}}^{S_n} \left(\mathcal{R}[S_{n_1}]_{|\pi_{S_{n_1}}^{-1}(0)} \boxtimes \dots \boxtimes \mathcal{R}[S_{n_k}]_{|\pi_{S_{n_k}}^{-1}(0)} \right)$$

as S_n -equivariant vector bundles. This confirms the conjectured factorization given in [EG, 11.27].

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REFERENCES

- [BE] **R. Bezrukavnikov** and **P. Etingof**, *Induction and restriction functors for rational Cherednik algebras*, *arXiv:08033639*.
- [E] **D. Eisenbud**, *Commutative Algebra With a View Toward Algebraic Geometry*, Springer-Verlag, New York, (1994).
- [EG] **P. Etingof** and **V. Ginzburg**, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphisms*, *Invent. Math* 147, 243-348 (2002).
- [KKS] **D. Kazhdan**, **B. Kostant** and **S. Sternberg** *Hamiltonian group actions and dynamical systems of Calogero type*, *Comm. Pure Appl. Math.*, 31, 481-507 (1978).
- [S1] **R. Steinberg** *Invariants of finite reflection groups*, *Canadian Journal of Mathematics*, 12, 616-618 (1960).
- [S2] **R. Steinberg** *Differential equations invariant under finite reflection groups*, *Transactions of the American Mathematical Society*, 112, 392-400 (1964).
- [W] **G. Wilson**, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, *Invent. Math* 133, 1-41 (1998).

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